Computation and Physics in Algebraic Geometry

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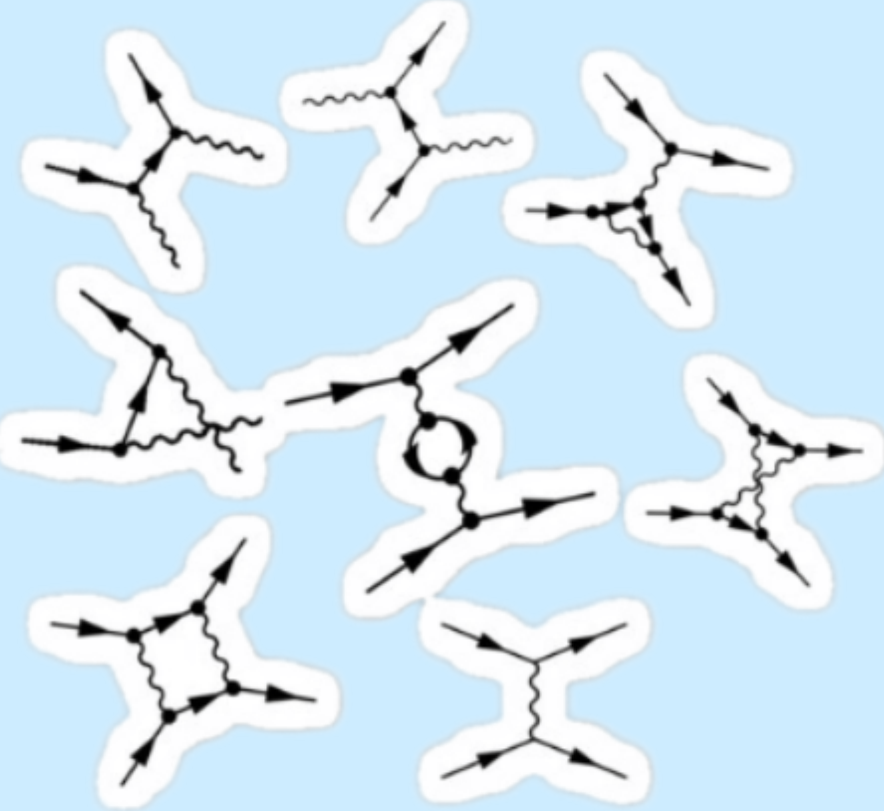
MAX PLANCK INSTITUTE
FOR MATHEMATICS IN THE SCIENCES

Public defense
Leipzig University, June 21, 2023

Part 1: Integrable systems and algebraic curves

$$\frac{\partial}{\partial x}\left(4p_t - 6pp_x - p_{xxx}\right) \;=\; 3p_{yy}$$

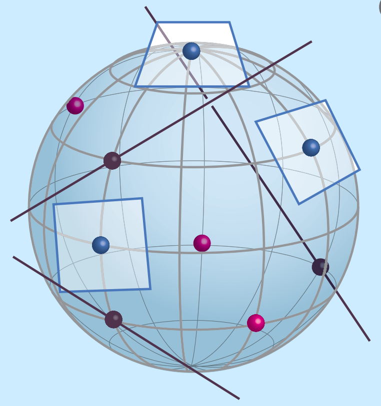
Part 2: Particle physics and very affine varieties

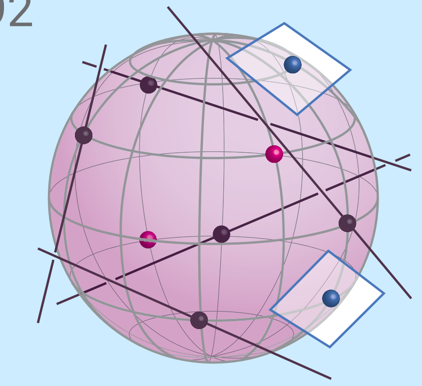


$$\int_\Gamma f^s\, x^\nu \,\frac{\mathrm{d}x}{x}$$

Part 3: Computation with quadrics

1 3 9 17 21 21 17 9 3 1
2 6 18 34 42 34 18 6 2
4 12 36 68 68 36 12 4
8 24 72 **104** 72 24 8
16 48 112 112 48 16
32 80 **128** 80 32
56 104 104 56
80 104 80
92 92
92





$$\begin{pmatrix} 0 & a & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & b & 1 \end{pmatrix}$$

Part 1: Integrable systems and algebraic curves

- **KP Solitons from Tropical Limits** (2023). Daniele Agostini, Claudia Fevola, Yelena Mandelshtam, and Bernd Sturmfels. *Journal of Symbolic Computation* **114** 282-301.

- **Hirota Varieties and Rational Nodal Curves** (2022). Claudia Fevola and Yelena Mandelshtam. ArXiv:2203.00203. Submitted to *Journal of Symbolic Computation.*

The Schottky problem: characterising Jacobian varieties of genus g curves

among all abelian varieties of dimension g

$\{\text{smooth curves of genus } g\}/_{\sim}$

$$\mathcal{M}_g \longrightarrow \mathcal{A}_g$$

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$$\mathscr{M}_g \longrightarrow \mathscr{A}_g$$

$$X \longmapsto \mathrm{Jac}(X) = \mathbb{C}^g/(\mathbb{Z}^g + B \cdot \mathbb{Z}^g)$$

Riemann matrix

$$B \in \mathfrak{H}_g = \{B \in \mathbb{S}^g(\mathbb{C}) \mid \mathrm{im}(B) \succeq 0\}$$

Siegel upper half space

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Siegel upper half space

$$\dim(\mathscr{M}_g) = 3g - 3 = 3,6,9,12,\ldots$$

$$\dim(\mathscr{A}_g) = \binom{g+1}{2} = 3,6,10,15,\ldots$$

Farkas, Grushevsky, Igusa, Salvati Manni, ….

Definition: The Riemann theta function is the complex analytic function

$$\theta(\mathbf{z} \mid B) \;=\; \sum_{\mathbf{c} \in \mathbb{Z}^g} \exp\left[\pi i \mathbf{c}^T B \mathbf{c} \;+ 2\pi i \mathbf{c}^T \mathbf{z}\right], \qquad \begin{array}{l} \mathbf{z} \in \mathbb{C}^g \\ B \in \mathfrak{H}_g \end{array}$$

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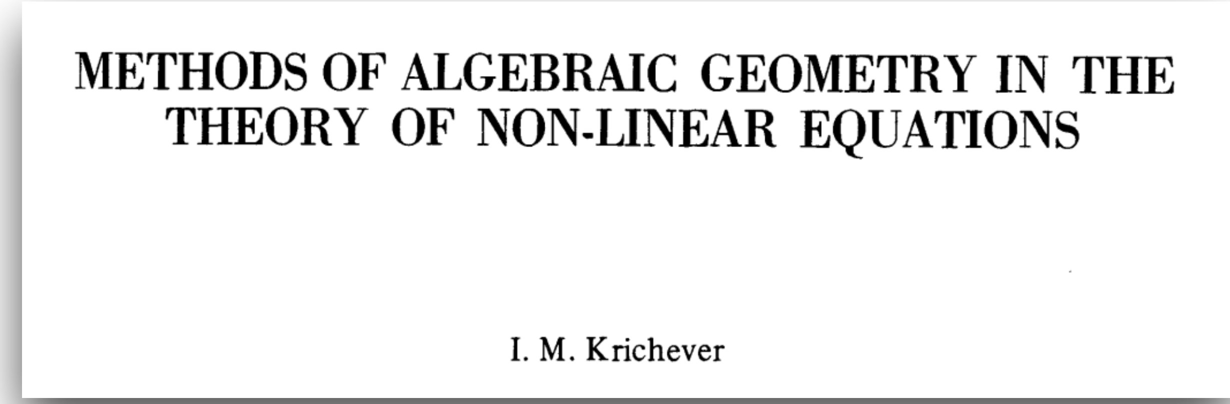
Theorem (Krichever - Shiota):

A matrix $B \in \mathfrak{H}_g$ comes from a Jacobian if and only if there are vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^g, \mathbf{u} \neq 0$ such that the function

$$p(x,y,t) \;=\; 2\partial_x^2 \log \theta(\mathbf{u}x + \mathbf{v}y + \mathbf{w}t + \mathbf{d} \mid B)$$


METHODS OF ALGEBRAIC GEOMETRY IN THE THEORY OF NON-LINEAR EQUATIONS

I. M. Krichever

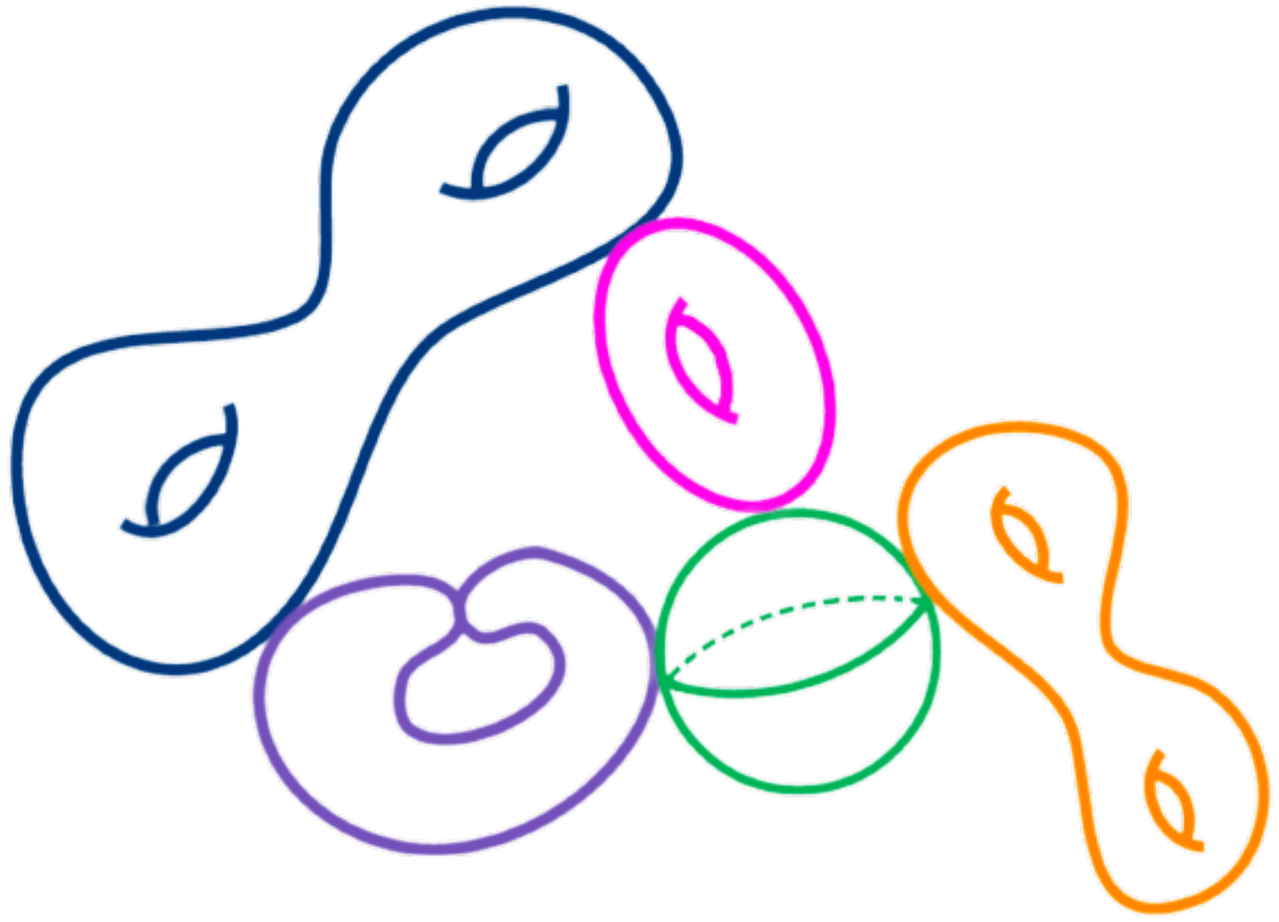
gives a solution to the KP equation

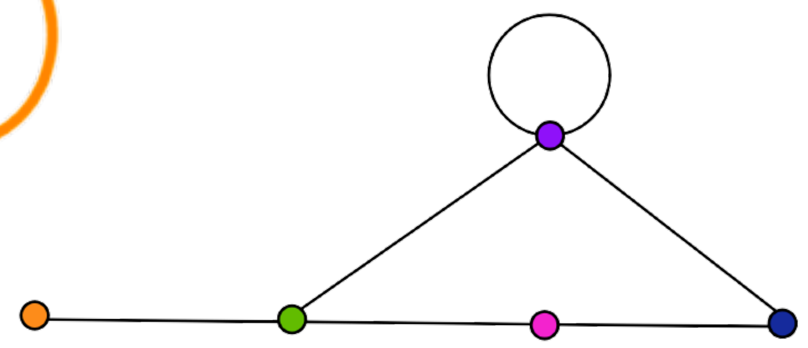
$$\frac{\partial}{\partial x}\left(4p_t - 6pp_x - p_{xxx}\right) \;=\; 3p_{yy}$$

for any $\mathbf{d} \in \mathbb{C}^g$.

Let X be a curve defined over a non-archimedean field $\mathbb{K}$, i.e. $\mathbb{Q}(\epsilon), \mathbb{C}\{\!\{\epsilon\}\!\}$

$$\epsilon \to 0$$





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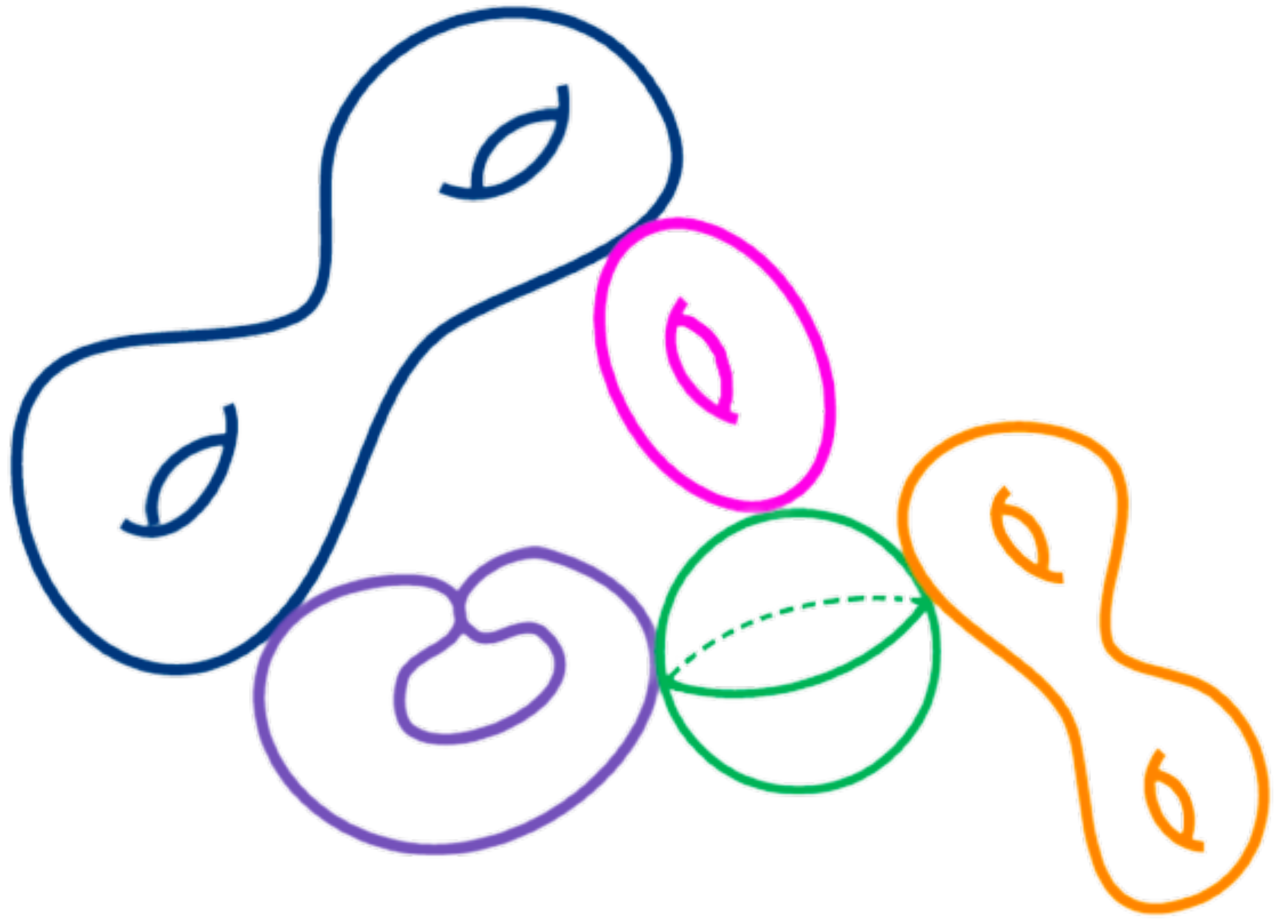
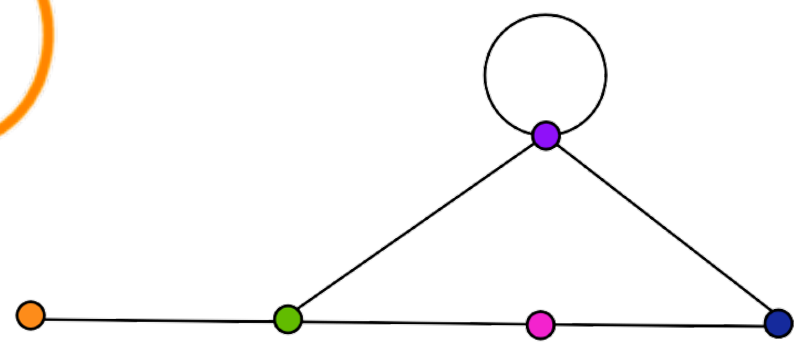
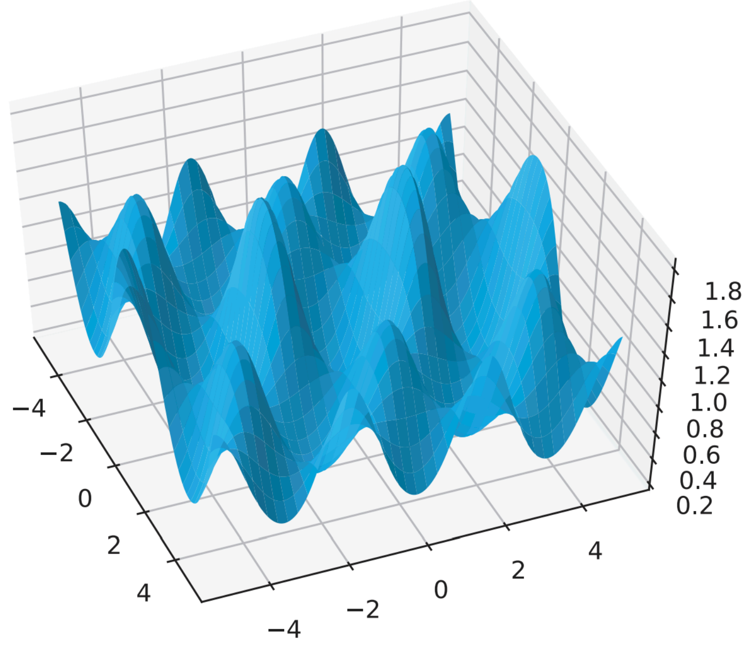
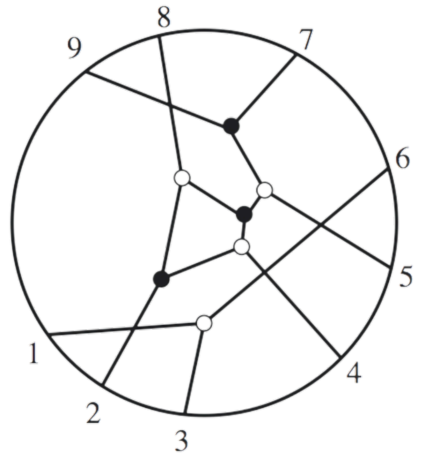
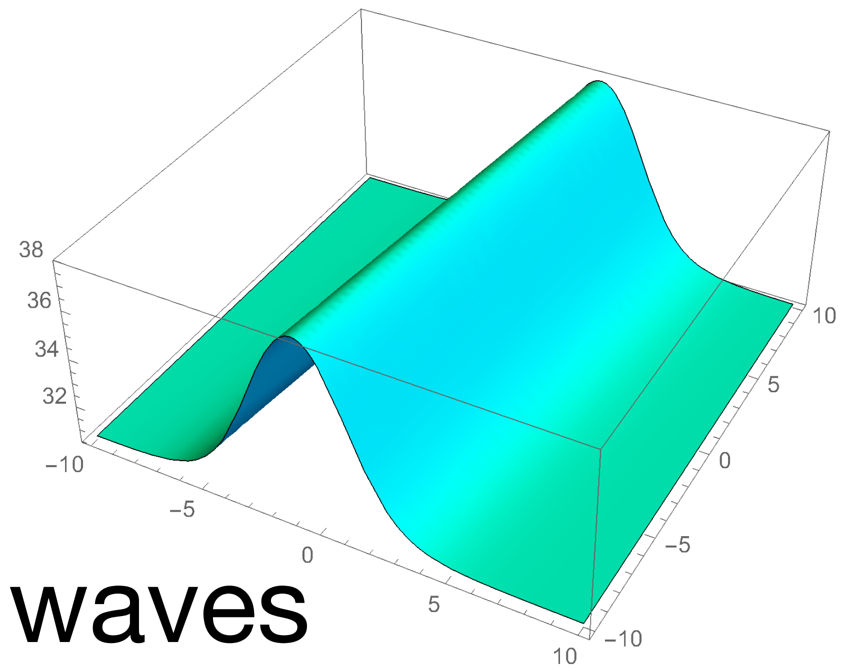




$$\theta(\mathbf{z}_\varepsilon \mid B_\varepsilon) = \sum_{\mathbf{c}\in\mathbb{Z}^g} \exp\left[\frac{1}{2}\mathbf{c}^T B_\varepsilon \mathbf{c} + \mathbf{c}^T \mathbf{z}_\varepsilon\right] \longrightarrow \theta_{\mathscr{C}}(\mathbf{z}) = \sum_{\substack{\mathbf{c}\in\mathscr{C}\subset\mathbb{Z}^g \\ |\mathscr{C}| = m < \infty}} a_{\mathbf{c}} \exp[\mathbf{c}^T \mathbf{z}]$$







Soliton waves

To each oriented metric graph $\Gamma = (V, E)$ we assign its tropical Riemann matrix Q

$$H_1(\Gamma, \mathbb{Z}) = \langle \gamma_1, \ldots, \gamma_g \rangle$$

$$L := \begin{bmatrix} \cdots & \gamma_1 & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & \gamma_g & \cdots \end{bmatrix} \in \mathbb{Z}^{g \times |E|}$$

$$D := \begin{bmatrix} l_1 & & \\ & \ddots & \\ & & l_{|E|} \end{bmatrix} \in \mathbb{Z}^{|E| \times |E|}$$

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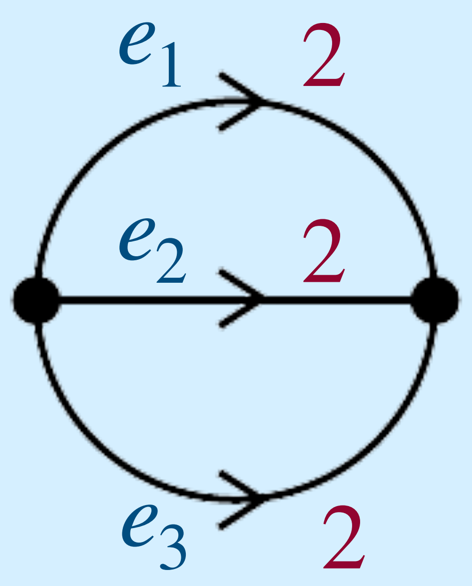
Example Let $X = V(y^2 - f(x))$

$$f(x) = (x-1)(x-1-\epsilon)(x-2)(x-2-\epsilon)(x-3)(x-3-\epsilon)$$



$$L := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \in \mathbb{Z}^{2 \times 3}$$

$$D := \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix} \in \mathbb{Z}^{3 \times 3}$$

$$Q := \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} \in \mathbb{Z}^{2 \times 2}$$

Let $\mathbf{a} \in \mathbb{R}^g$, $\mathbf{z}_\epsilon = \mathbf{z} + \frac{1}{\epsilon} Q\mathbf{a}$, $B_\epsilon = -\frac{1}{\epsilon} Q + R(\epsilon)$

$$\lim_{\epsilon \to 0} \theta(\mathbf{z}_\epsilon, B_\epsilon) = \lim_{\epsilon \to 0} \sum_{\mathbf{c} \in \mathbb{Z}^g} \exp\left[-\frac{1}{2\epsilon}\mathbf{c}^T Q \mathbf{c} + \frac{1}{\epsilon}\mathbf{c}^T Q \mathbf{a}\right] \cdot \exp\left[\frac{1}{2}\mathbf{c}^T R(\epsilon)\mathbf{c} + \mathbf{c}^T \mathbf{z}\right]$$

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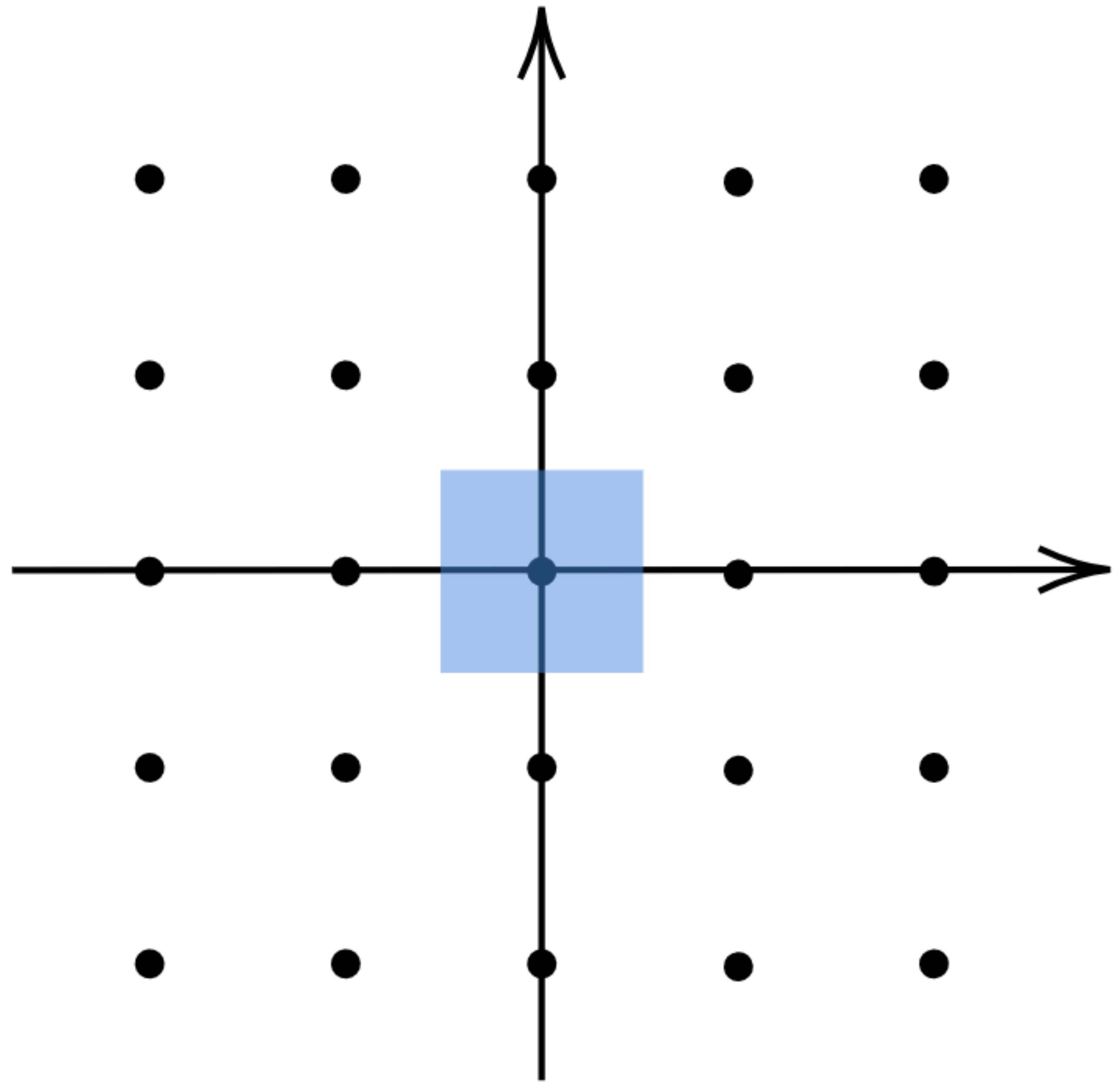
$\mathbf{a}^T Q \mathbf{a} \leq (\mathbf{a} - \mathbf{c})^T Q (\mathbf{a} - \mathbf{c})$ for all $\mathbf{c} \in \mathbb{Z}^g$



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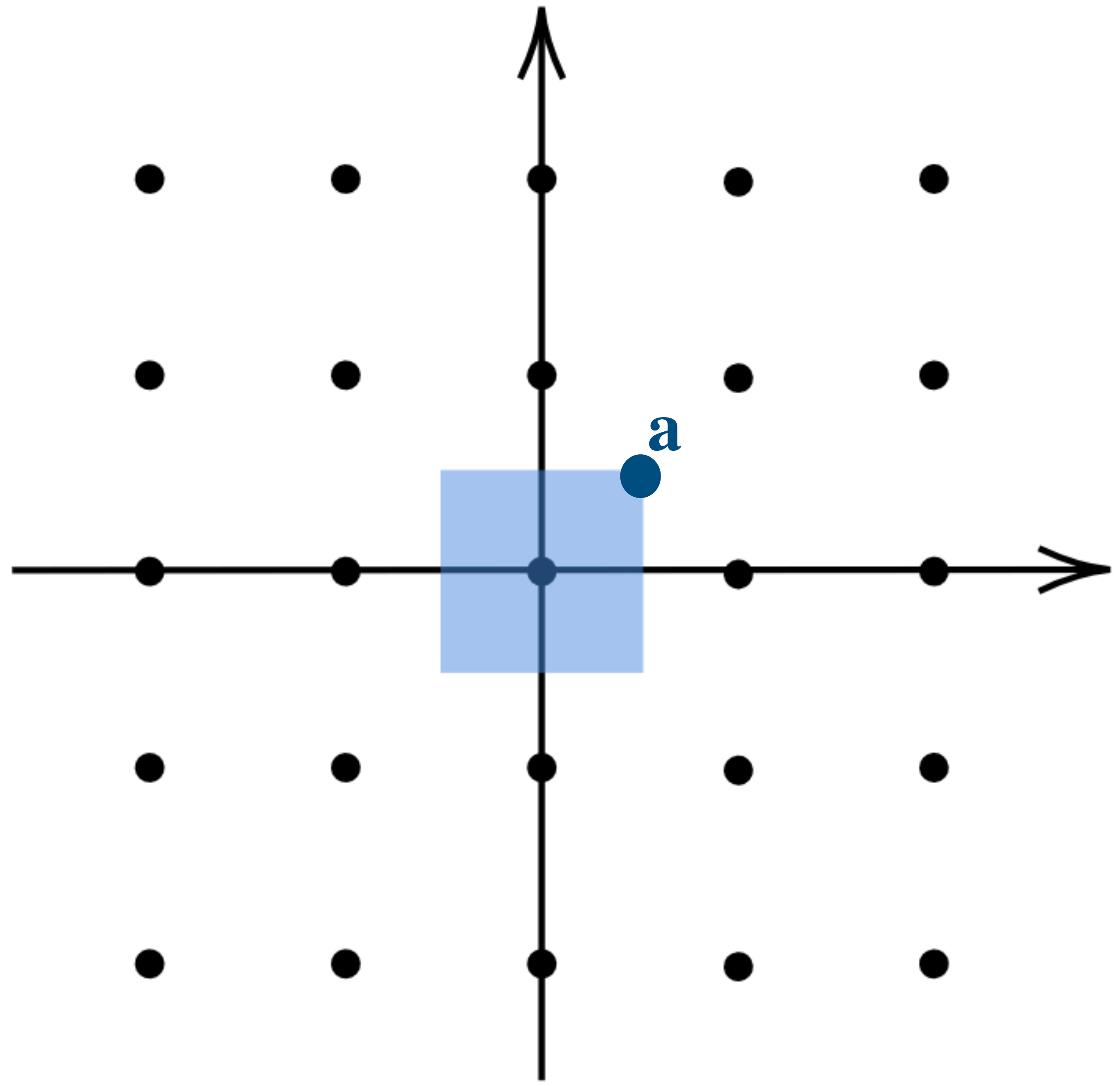

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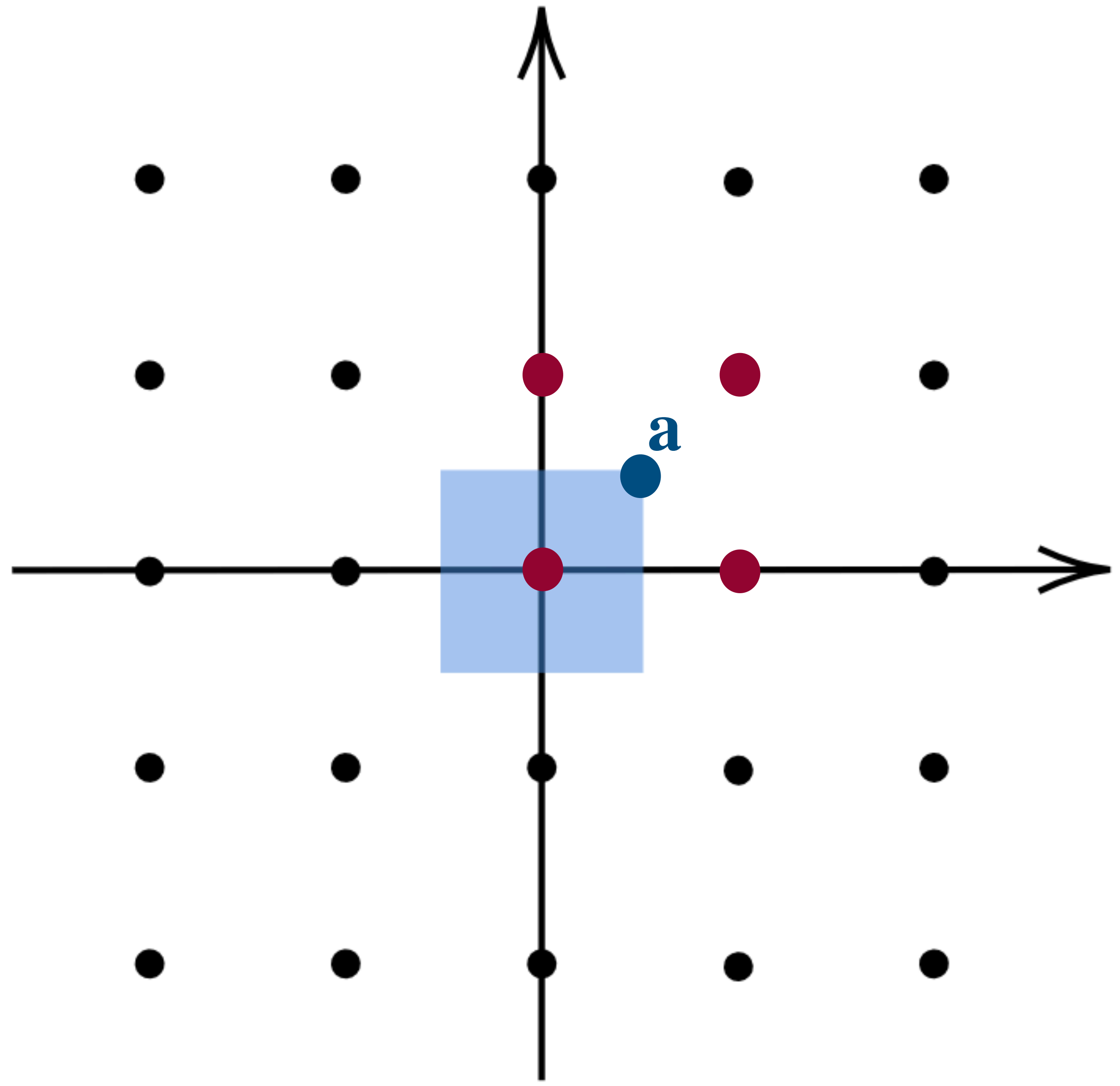
$$\mathscr{D}_{\mathbf{a},Q} := \{ \mathbf{c} \in \mathbb{Z}^g \, : \, \mathbf{a}^T Q \mathbf{a} = (\mathbf{a} - \mathbf{c})^T Q (\mathbf{a} - \mathbf{c}) \}$$



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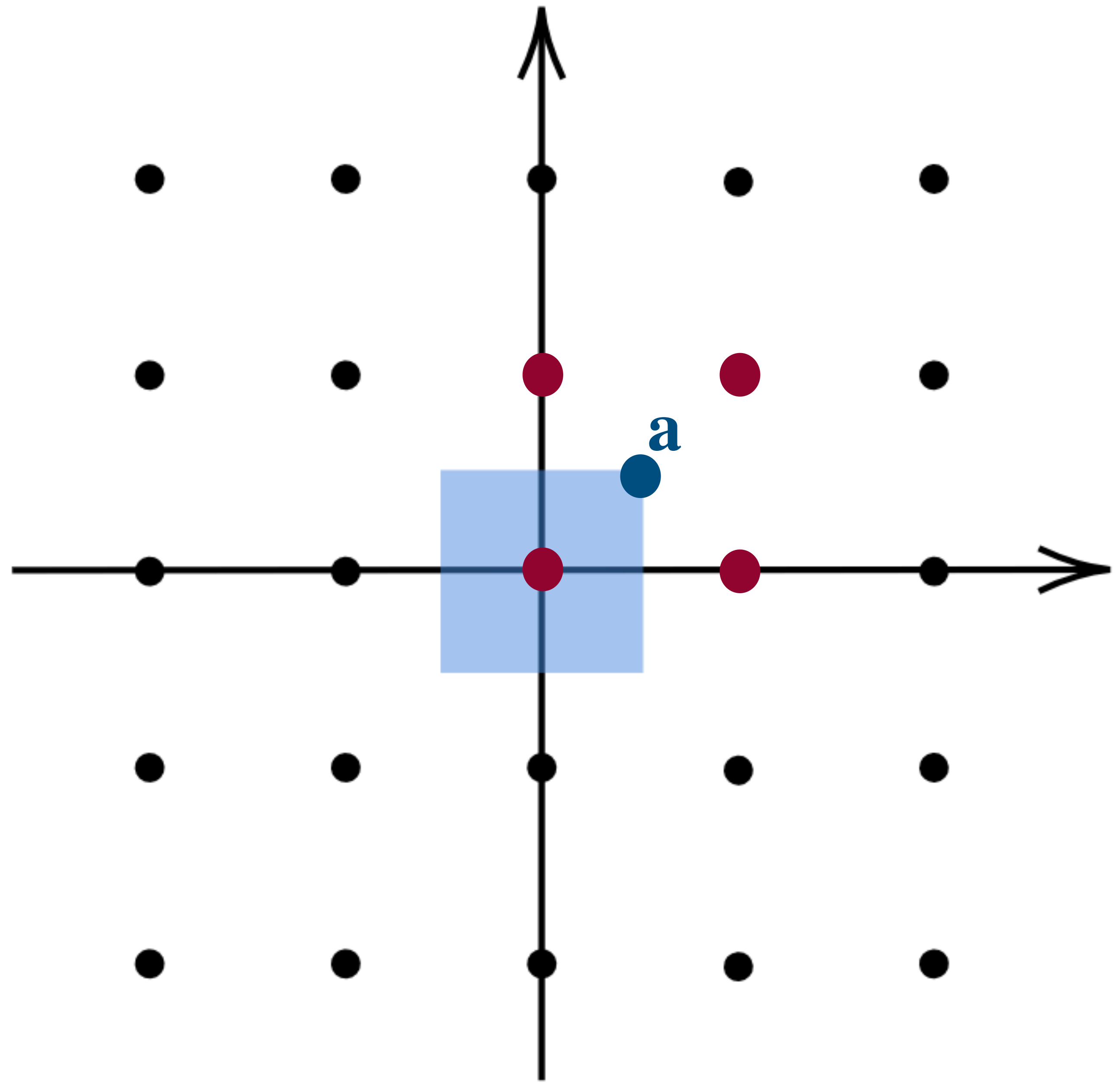

Theorem (Agostini, F., Mandelshtam, Sturmfels):
Fix $\mathbf{a}$ in the Voronoi cell of the tropical Riemann matrix Q, then

$$\lim_{\epsilon\to 0} \theta(\mathbf{z}_\epsilon, B_\epsilon) = \sum_{\mathbf{c}\in\mathscr{C}} a_{\mathbf{c}} \exp[\mathbf{c}^T\mathbf{z}]$$

where

$$a_{\mathbf{c}} = \exp\left[\frac{1}{2}\mathbf{c}^T R(0)\mathbf{c}\right]$$

$$\mathscr{C} = \mathcal{D}_{\mathbf{a},Q}$$

Definition: The Hirota variety $\mathscr{H}_{\mathscr{C}}$ consists of all points $(\mathbf{a}, (\mathbf{u}, \mathbf{v}, \mathbf{w})) \in (\mathbb{K}^*)^m \times \mathbb{WP}^{3g-1}$ such that $\theta_{\mathscr{C}}(\mathbf{u}x + \mathbf{v}y + \mathbf{w}t)$ gives a solution to the KP equation.

$$\mathscr{H}^I_{\mathscr{C}} := \overline{\{(\mathbf{a}, (\mathbf{u}, \mathbf{v}, \mathbf{w})) \in \mathscr{H}_{\mathscr{C}} \,:\, \mathbf{u} \neq \mathbf{0}\}}$$

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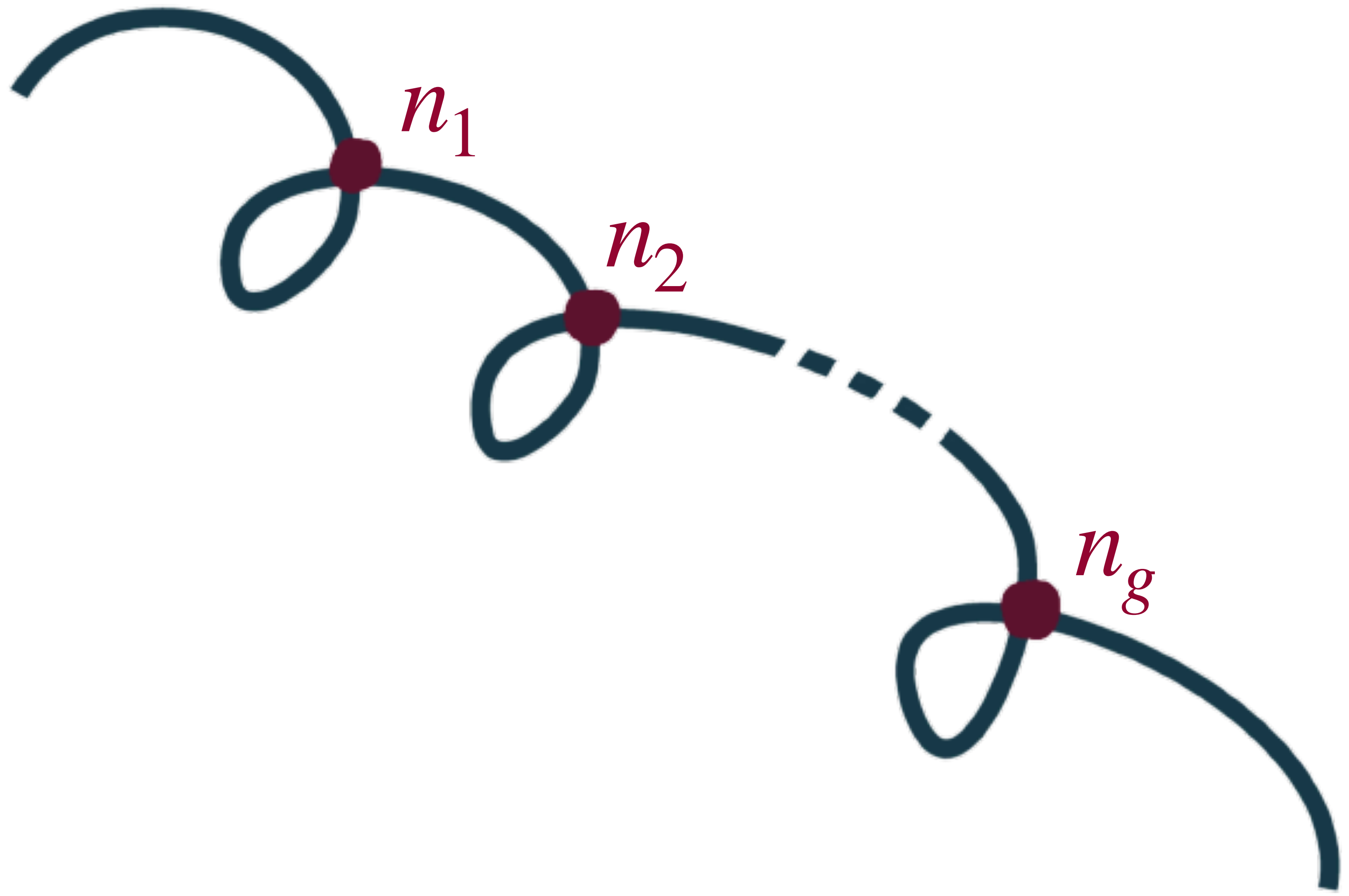
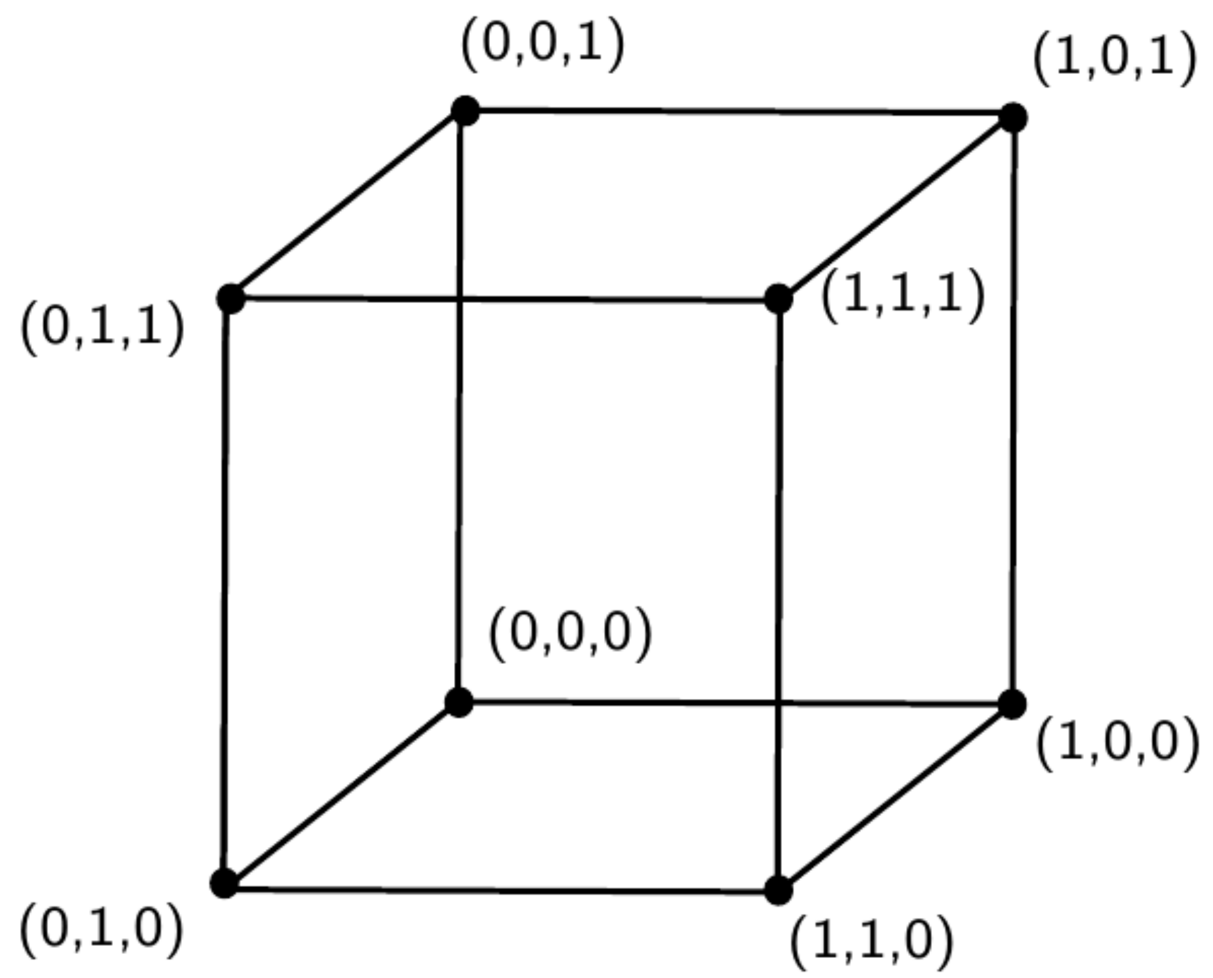
Remark: Studying $\mathscr{H}^I_{\mathscr{C}}$ provides a new approach to the Schottky problem for nodal curves → **Combinatorial Schottky problem**





Part 2: Particle physics and very affine varieties

- **Vector Spaces of Generalized Euler Integrals** (2022). Daniele Agostini, Claudia Fevola, Anna-Laura Sattelberger, and Simon Telen. ArXiv:2208.08967. Submitted to *Communications in Number Theory and Physics.*
- **Likelihood Degenerations** (2023). Daniele Agostini, Taylor Brysiewicz, Claudia Fevola, Lukas Kühne, Bernd Sturmfels, and Simon Telen. *Advances in Mathematics* **414** 108863.

Generalised Euler Integrals [GKZ]

$$\int_\Gamma f^s\, x^\nu \,\frac{\mathrm{d}x}{x} \;=\; \int_\Gamma \left(\prod_{j=1}^{\ell} f_j^{s_j}\right)\cdot\left(\prod_{i=1}^{n} x_i^{\nu_i}\right)\frac{\mathrm{d}x_1}{x_1}\wedge\cdots\wedge\frac{\mathrm{d}x_n}{x_n}$$

- $x = (x_1, \ldots, x_n) \in (\mathbb{C}^*)^n$
- $f = (f_1, \ldots, f_\ell) \in \mathbb{C}[x, x^{-1}]^\ell$
- $s = (s_1, \ldots, s_\ell) \in \mathbb{C}^\ell, \quad \nu = (\nu_1, \ldots, \nu_n) \in \mathbb{C}^n$
- $\Gamma \in H_n(X, \omega)$, where $\omega = \mathrm{d}log(f^s x^\nu)$

$$X \;:=\; \{\, x \in (\mathbb{C}^*)^n \mid f_1(x)\cdots f_\ell(x) \neq 0 \,\} \;=\; (\mathbb{C}^*)^n \setminus V(f_1\cdots f_\ell)$$

Generalised Euler Integrals [GKZ]

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Feynman integrals: $\ell = 1$, $f =$ Graph polynomial

Appendix: Feynman integrals for mathematicians

$$X := \{\, x \in (\mathbb{C}^*)^n \mid f_1(x) \cdots f_\ell(x) \neq 0 \,\} = (\mathbb{C}^*)^n \setminus V(f_1 \cdots f_\ell)$$

Vector spaces of Generalised Euler Integrals

$$V_\Gamma := \mathrm{Span}_{\mathbb{C}} \left\{ [\Gamma] \longmapsto \int_\Gamma f^{s+a}\, x^{\nu+b}\, \frac{\mathrm{d}x}{x} \right\}_{(a,b) \in \mathbb{Z}^\ell \times \mathbb{Z}^n}$$

$\rightarrow$ Twisted (co)homology

Mastrolia, Mizera

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→ Twisted (co)homology

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$$V_{s,\nu} := \mathrm{Span}_{\mathbb{C}(s,\nu)} \left\{ (s,\nu) \longmapsto \int_\Gamma f^{s+a}\, x^{\nu+b} \frac{\mathrm{d}x}{x} \right\}_{(a,b)\in\mathbb{Z}^\ell\times\mathbb{Z}^n}$$

Difference operators and Mellin transform ←

Bitoun, Bogner, Klausen, Panzer

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Difference operators
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$$V_{c^*} := \mathrm{Span}_{\mathbb{C}}\left\{c \longmapsto \int_\Gamma f(x;c)^s\, x^{\nu}\, \frac{\mathrm{d}x}{x}\right\}_{[\Gamma]\in H_n(X,\omega)}$$

→ GKZ systems

Matsubara-Heo, Chestnov, …

Theorem (Agostini, F., Sattelberger, Telen):

Let $f = (f_1, \ldots, f_\ell) \in \mathbb{C}[x, x^{-1}]^\ell$ be Laurent polynomials with fixed monomial supports and generic coefficients. Consider $V_\Gamma, V_{s,\nu}, V_{c^*}$ with generic choices of parameters each. Then

$$\dim_{\mathbb{C}}(V_\Gamma) \;=\; \dim_{\mathbb{C}(s,\nu)} V_{s,\nu} \;=\; \dim_{\mathbb{C}}(V_{c^*}) \;=\; (-1)^n \cdot \chi(X)\,.$$

Topological Euler characteristic

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Topological Euler characteristic

Computing Euler characteristics

Theorem (Huh): $|\chi(X)|$ equals the number of critical points of

$$L \;=\; \log(f^s x^\nu) \;=\; \sum_{j=1}^{\ell} s_j \log f_j + \sum_{i=1}^{n} \nu_i \log x_i$$

for general s, ν.

Solving rational function equations using HomotopyContinuation.jl

```
using HomotopyContinuation
@var x y s ν[1:2]
f = -x*y^2 + 2*x*y^3 + 3*x^2*y - x^2*y^3 - 2*x^3*y + 3*x^3*y^2
L = s*log(f) + ν[1]*log(x) + ν[2]*log(y)
F = System(differentiate(L,[x;y]), parameters = [s;ν])
monodromy_solve(F)
```

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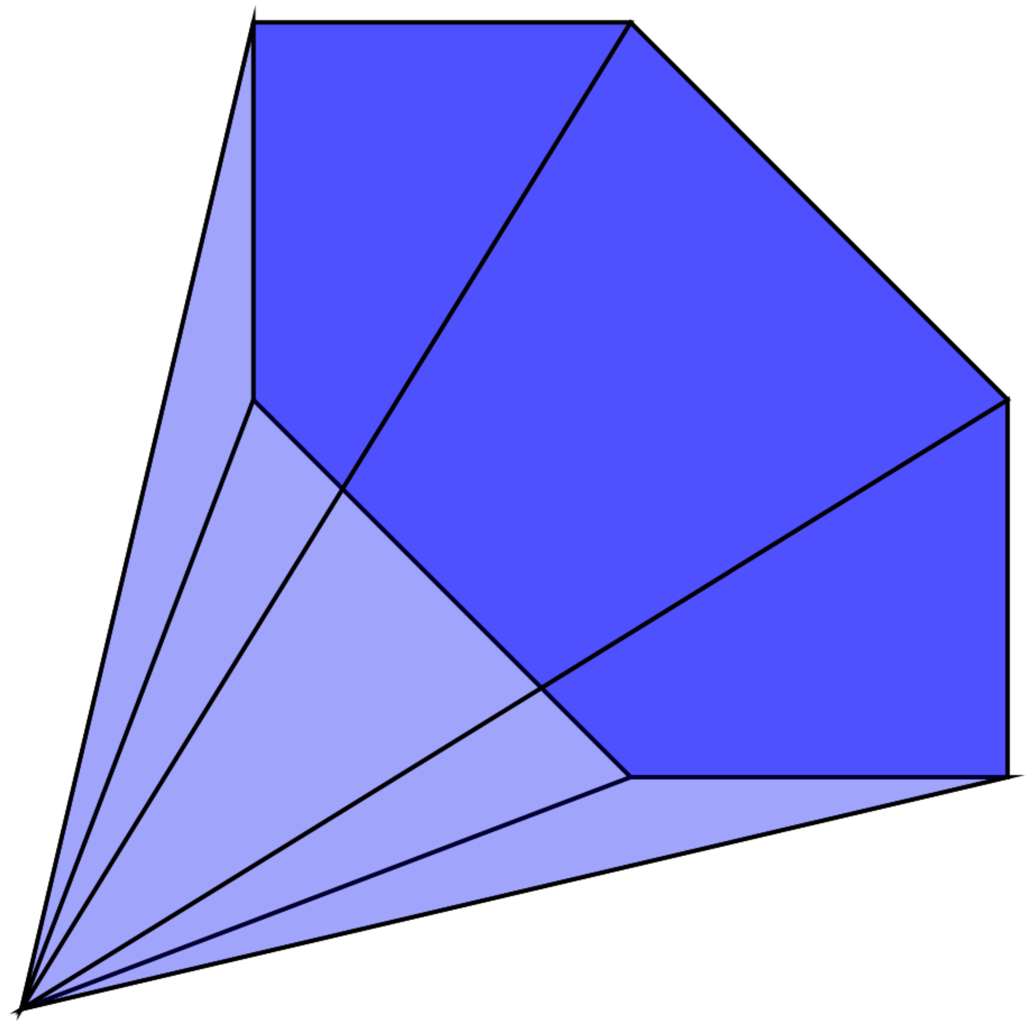
monodromy_solve(F)
```

For generic choices of the parameters:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 3 & 1 & 3 & 1 & 2 \end{pmatrix} \rightarrow$$



$$\boxed{\mathrm{vol}(\mathrm{Conv}(A)) = \chi(X) = 6}$$

Example:

$$X = X(3,7) = \mathrm{Gr}(3,7)^{\circ}/(\mathbb{C}^*)^7$$

Moduli space of 7 points in $\mathbb{P}^2$ in linearly general position

$$M_{3,7} = \begin{array}{c} \begin{pmatrix} 0 & 0 & -1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_5 & x_6 & x_7 \\ -1 & 0 & 0 & 1 & y_5 & y_6 & y_7 \end{pmatrix} \\ \begin{array}{ccccccc} & & i & & j & & k \end{array} \end{array}$$

$$\ell = \binom{7}{3} = 35, n = 6 \qquad \int_{\Gamma} \left(\prod_{1 \le i < j < k \le 7} p_{ijk}(x,y)^{s_{ijk}+a_{ijk}} x^{\nu_1 + b_1} y^{\nu_2 + b_2} \right) \frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y}$$

HomotopyContinuation.jl

$$\chi(X(3,7)) = 1272$$

Regular Article - Theoretical Physics | Open Access | Published: 28 May 2020

Singular solutions in soft limits

Freddy Cachazo, Bruno Umbert & Yong Zhang

Journal of High Energy Physics **2020**, Article number: 148 (2020) | Cite this article

Theorem (ABFKST):

m	4	5	6	7	8	9
$\lvert\chi(X(3,m))\rvert$	1	2	26	1 272	188 112	74 570 400

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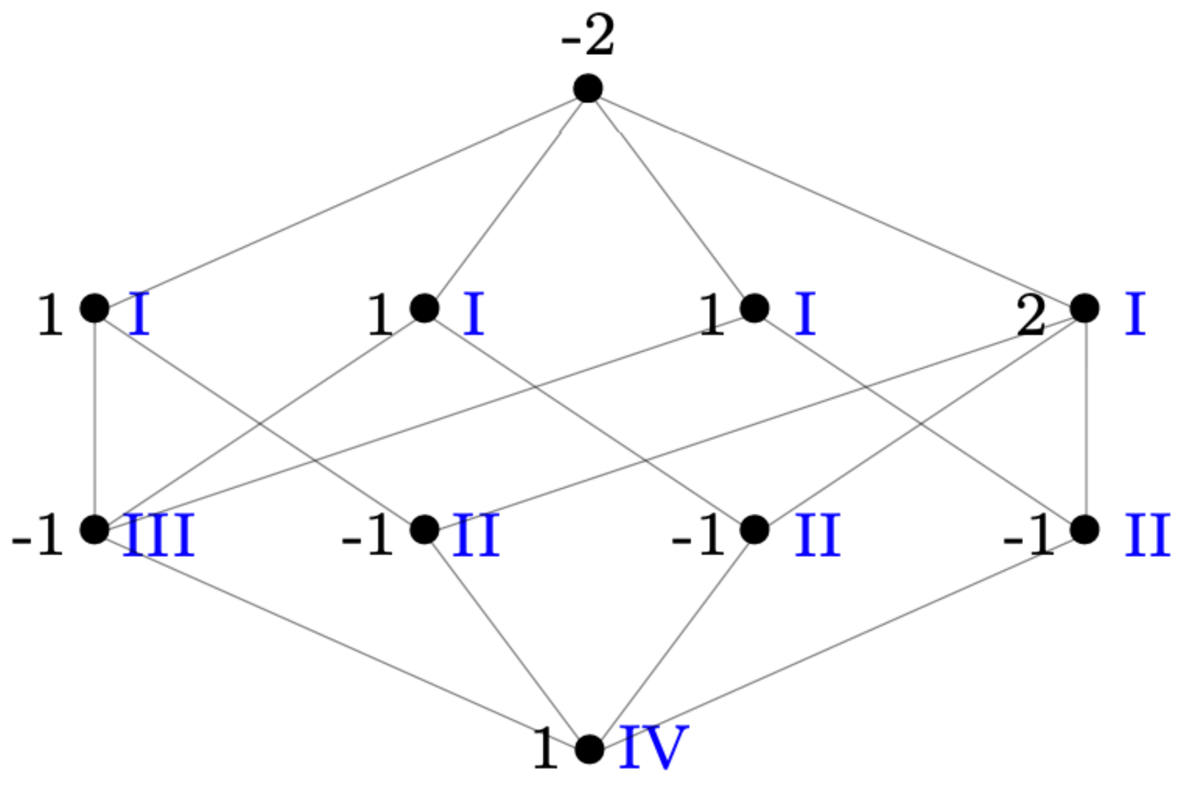
Idea of proof:

$$\pi_{k,m} \; : \; X(k, m+1) \longrightarrow X(k,m)$$

$\pi_{k,m}$ fibration: $\chi(X(k,m+1)) = \chi(F) \cdot \chi(X(k,m))$ True for $k = 2$!

For $k \geq 3$, $\pi_{k,m}$ is a stratified fibration:

$$\chi(X(k,m+1)) = \chi(F) \cdot \chi(X(k,m)) \; + \ldots$$



Part 3: Computation with quadrics

Linear Spaces of Symmetric Matrices

A collaboration project at MPI Leipzig and worldwide

- **Pencils of Quadrics: Old and New** (2021). Claudia Fevola, Yelena Mandelshtam, and Bernd Sturmfels. Le *Matematiche* **76** 319-335.

- **Tangent Quadrics in Real 3-Space** (2021). Taylor Brysiewicz, Claudia Fevola, and Bernd Sturmfels. *Le Matematiche* **76** 355-367.

Part 3: Computation with quadrics

Applications: LSSM are

- **Linear Gaussian models** in algebraic statistics;
- **Quadrics** in enumerative algebraic geometry;
- Spectrahedra in optimization;
- Tensors in nonlinear algebra.

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Pencils of quadrics are two-dimensional linear subspaces in $\mathbb{S}^n$

$$\mathscr{L} \;=\; \{\, \lambda A + \mu B \;\mid\; \lambda, \mu \in \mathbb{C} \,\} \;\in\; \mathrm{Gr}(2, \mathbb{S}^n)$$

$A, B \in \mathbb{S}^n$

$\mathbf{x}^T A \mathbf{x}$

Quadric hypersurface in $\mathbb{P}^{n-1}$

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Two pencils are isomorphic if they lie in the same $\mathrm{GL}(n)$ – orbit

Each stratum is indexed by a Segre symbol σ

Segre symbols

Pieter Belmans

November 9, 2016

Abstract

The classification of (possibly singular) intersections of quadric hypersurfaces turns out to be completely classical, and it is the subject of the PhD thesis of Corrado Segre (of Segre embedding fame, not to be confused with his student Beniamino Segre). In this short note I will recall the definition, and introduce some of the competing terminology. I will also summarise the classification for $\mathbb{P}^2$ and $\mathbb{P}^3$.

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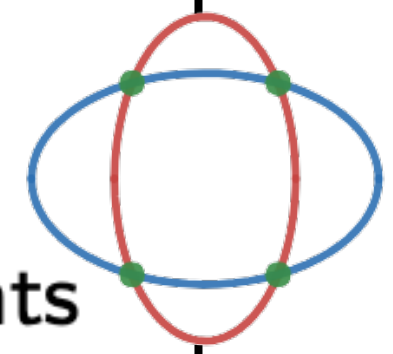
Segre symbol	quadrics P, Q	variety in $\mathbb{P}^2$
$[1,1,1]$	$ax^2+by^2+cz^2$ $x^2+y^2+z^2$	four reduced points
$[2,1]$	$2axy+y^2+cz^2$ $2xy+z^2$	one double point, two others
$[3]$	$2axz+ay^2+2yz$ $2xz+y^2$	one triple point, one other
$[(1,1),1]$	$a(x^2+y^2)+cz^2$ $x^2+y^2+z^2$	two double points
$[(2,1)]$	$2axy+y^2+az^2$ $2xy+z^2$	quadruple point



The **maximum likelihood estimation for Gaussians** is to maximize the value of the log-likelihood function

$$\boxed{\begin{aligned} \ell_S : \mathbb{S}^n_{>0} &\to \mathbb{R} \\ M &\mapsto \log(\det(M)) - \mathrm{trace}(SM) \end{aligned}}$$

The (reciprocal) ML degree of $\mathscr{L}$ is the number of complex critical points of ℓ_S on $\mathscr{L}$ ($\mathscr{L}^{-1}$) for generic $S \in \mathbb{S}^n$

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Theorem (F., Mandelshtam, Sturmfels):

Let $\mathscr{L}$ be a pencil of quadric with Segre symbol $\sigma = [\sigma_1, \ldots, \sigma_r]$. Then

$$\operatorname{mld}(\mathscr{L}) \;=\; r - 1 \qquad \text{and} \qquad \operatorname{rmld}(\mathscr{L}) \;=\; \sum_{i=1}^{r} \sigma_{i1} + r - 3$$

$n = 4$

Segre symbol	(mld, rmld)	quadrics P, Q	variety in $\mathbb{P}^3$
$[1,1,1,1]$	$(3,5)$	$ax^2+by^2+cz^2+du^2$ $x^2+y^2+z^2+u^2$	*elliptic curve*
$[2,1,1]$	$(2,4)$	$2axy+y^2+cz^2+du^2$ $2xy+z^2+u^2$	*nodal curve*
$[(1,1),1,1]$	$(2,3)$	$a(x^2+y^2)+cz^2+du^2$ $x^2+y^2+z^2+u^2$	*two conics meet twice*
$[3,1]$	$(1,3)$	$2axz+ay^2+2yz+du^2$ $2xz+y^2+u^2$	*cuspidal curve*
$[2,2]$	$(1,3)$	$2axy+y^2+2bzu+u^2$ $2xy+2zu$	*twisted cubic with secant*
$[(2,1),1]$	$(1,2)$	$2axy+y^2+az^2+du^2$ $2xy+z^2+u^2$	*two tangent conics*
$[4]$	$(0,2)$	$2axu+2ayz+2yu+z^2$ $2xu+2yz$	*twisted cubic with tangent*
$[2,(1,1)]$	$(1,2)$	$2axy+y^2+c(z^2+u^2)$ $2xy+z^2+u^2$	*conic meets two lines*
$[(3,1)]$	$(0,1)$	$2axz+ay^2+2yz+au^2$ $2xz+y^2+u^2$	*conic and two lines concur*
$[(1,1),(1,1)]$	$(1,1)$	$a(x^2+y^2)+c(z^2+u^2)$ $x^2+y^2+z^2+u^2$	*quadrangle of lines*
$[(1,1,1),1]$	$(1,1)$	$a(x^2+y^2+z^2)+du^2$ $x^2+y^2+z^2+u^2$	*double conic*
$[(2,2)]$	$(0,0)$	$2axy+y^2+2azu+u^2$ $2xy+2zu$	*double line and two lines*
$[(2,1,1)]$	$(0,0)$	$2axy+y^2+a(z^2+u^2)$ $2xy+z^2+u^2$	*two double lines*

Real tangent quadrics

How many smooth degree d hypersurfaces in $\mathbb{P}^n$ are tangent to $\binom{n+d}{d} - 1$ general linear spaces of various dimensions?

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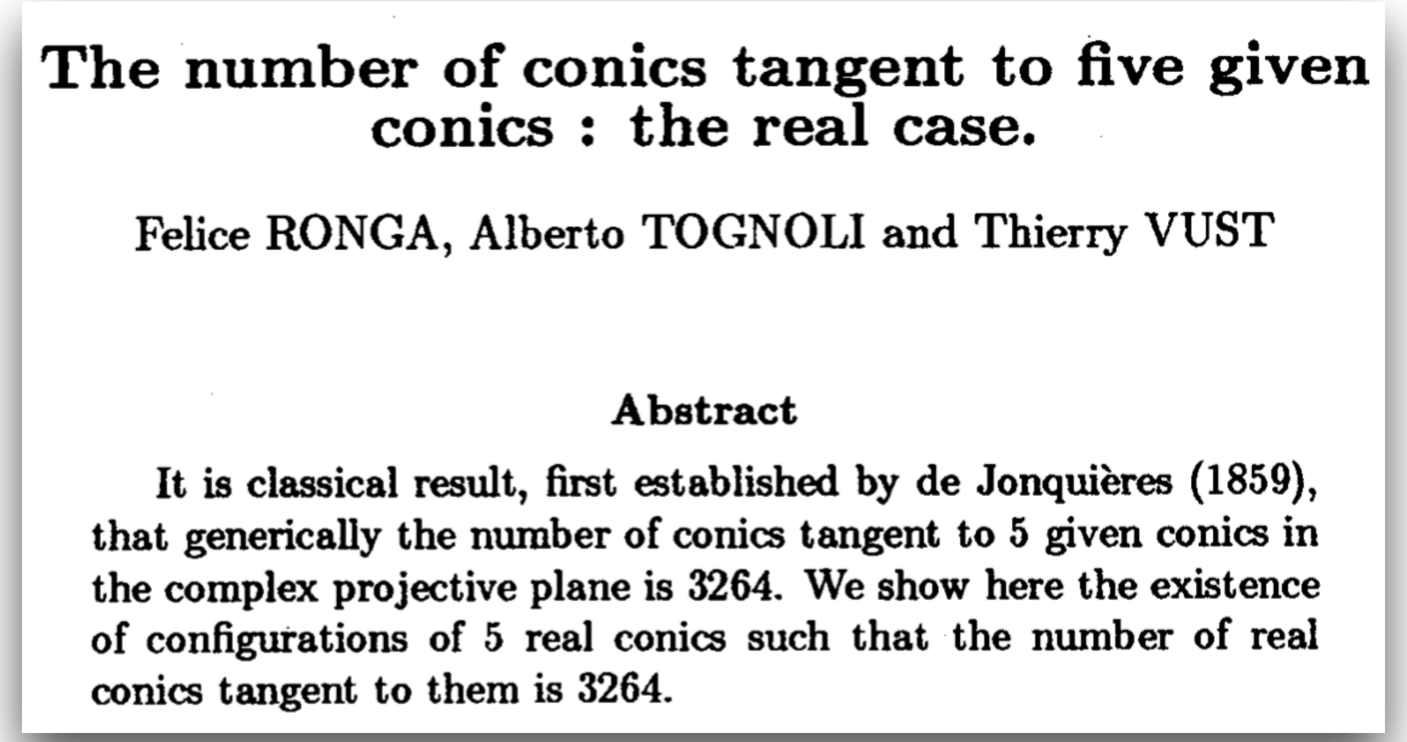
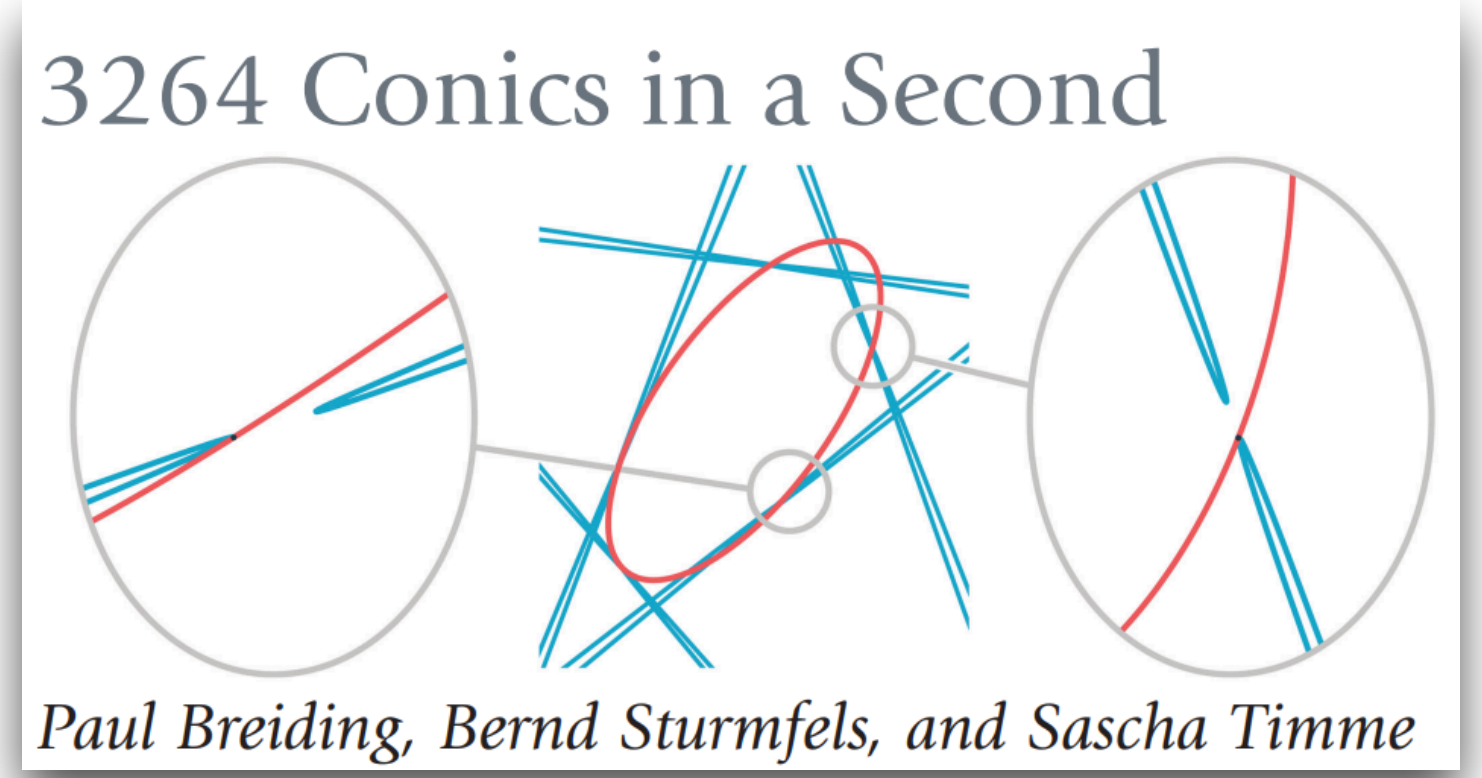


The number of conics tangent to five given conics : the real case.

Felice RONGA, Alberto TOGNOLI and Thierry VUST

Abstract

It is classical result, first established by de Jonquières (1859), that generically the number of conics tangent to 5 given conics in the complex projective plane is 3264. We show here the existence of configurations of 5 real conics such that the number of real conics tangent to them is 3264.



Real tangent quadrics

How many smooth degree d hypersurfaces in $\mathbb{P}^n$ are tangent to $\binom{n+d}{d}-1$ general linear spaces of various dimensions?

Schubert (1879): *There are $666\,841\,088$ quadrics in $\mathbb{P}^3$ that are tangent to 9 given quadrics*

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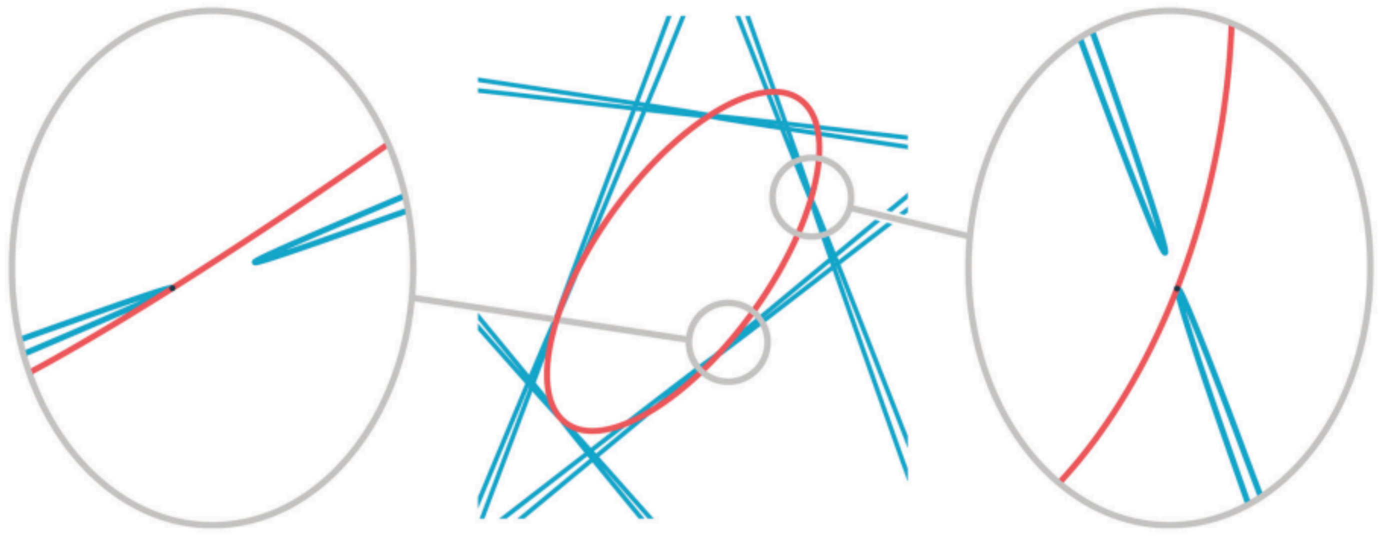
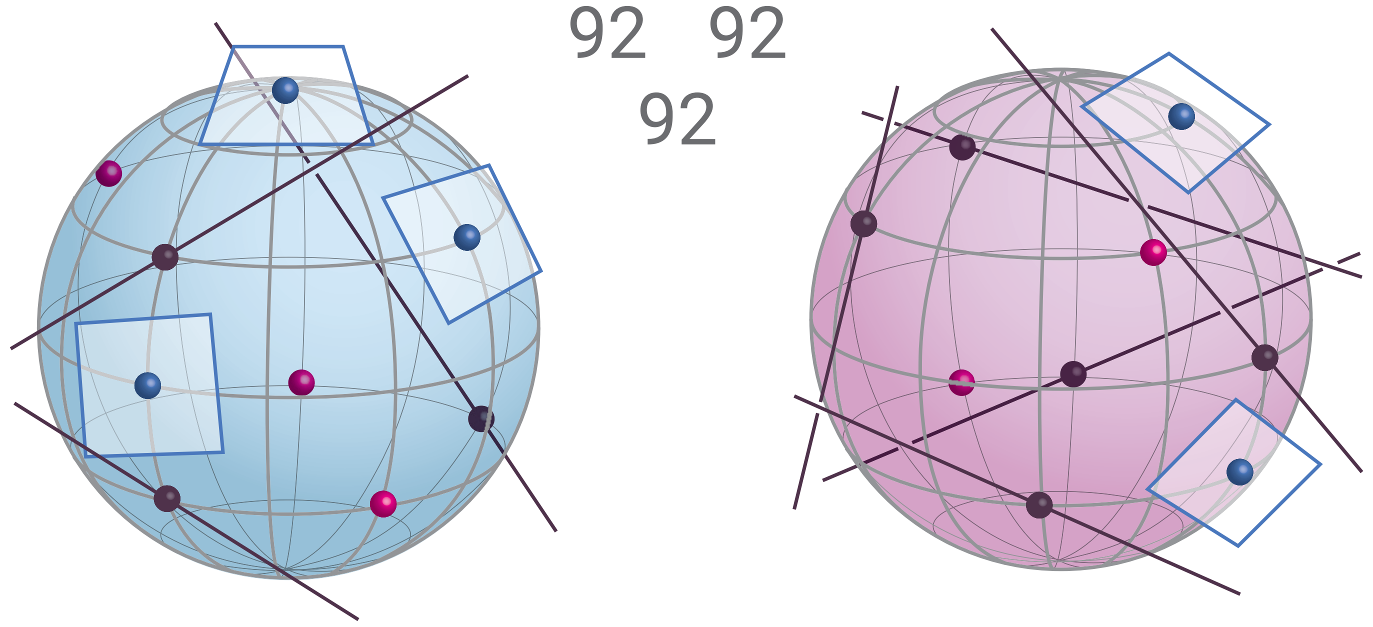
3264 Conics in a Second



Paul Breiding, Bernd Sturmfels, and Sascha Timme

1 3 9 17 21 21 17 9 3 1
2 6 18 34 42 34 18 6 2
4 12 36 68 68 36 12 4
8 24 72 **104** 72 24 8
16 48 112 112 48 16
32 80 **128** 80 32
56 104 104 56
80 104 80
92 92
92



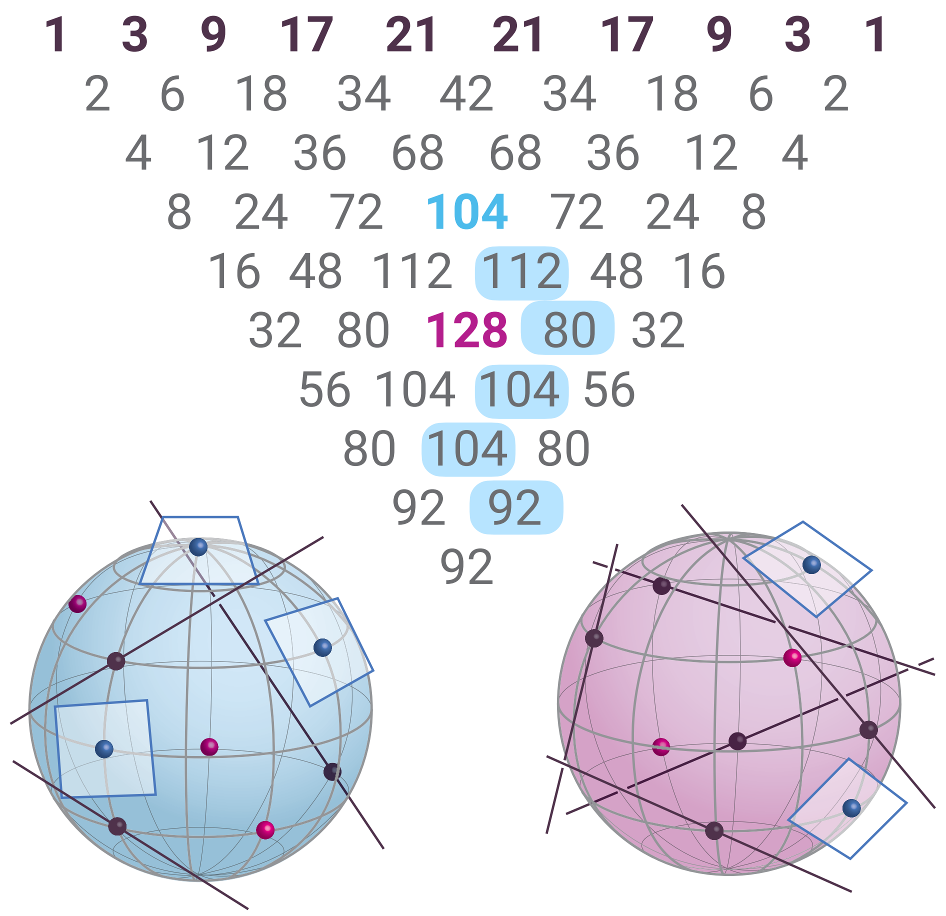
$$[2(p+\ell+h)]^9 = 2^9 \sum_{\alpha+\beta+\gamma=9} \frac{9!}{\alpha!\beta!\gamma!} p^\alpha \ell^\beta h^\gamma = 2^9(\cdots + 1680\cdot 104 + \cdots + 756 \cdot 128 + \cdots)$$

Theorem (Brysiewicz, F., Sturmfels):

For at least 46 of the 55 problems in Schubert's triangle, there exists an open set of real instances, consisting of α points, β lines and γ planes, such that all complex solutions in $\mathbb{P}^9$ to the polynomial equations

$$P_i X P_i^T = \det(L_j X L_j^T) = \det(H_k X H_k^T) = 0 \text{ for } 1 \leq i \leq \alpha, 1 \leq j \leq \beta, 1 \leq k \leq \gamma$$

are real.



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1 3 9 17 21 21 17 9 3 1
2 6 18 34 42 34 18 6 2
4 12 36 68 68 36 12 4
8 24 72 **104** 72 24 8
16 48 112 112 48 16
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56 104 104 56
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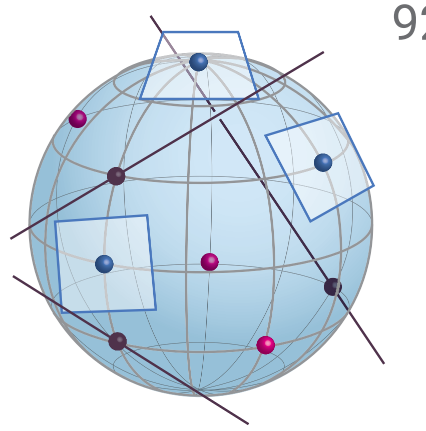
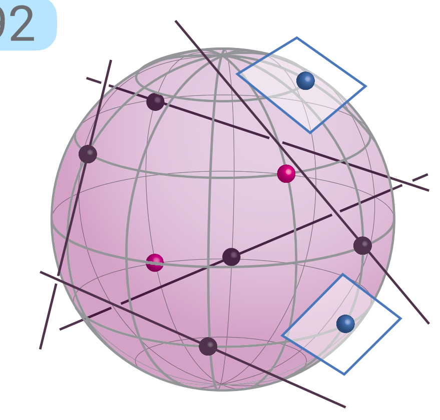
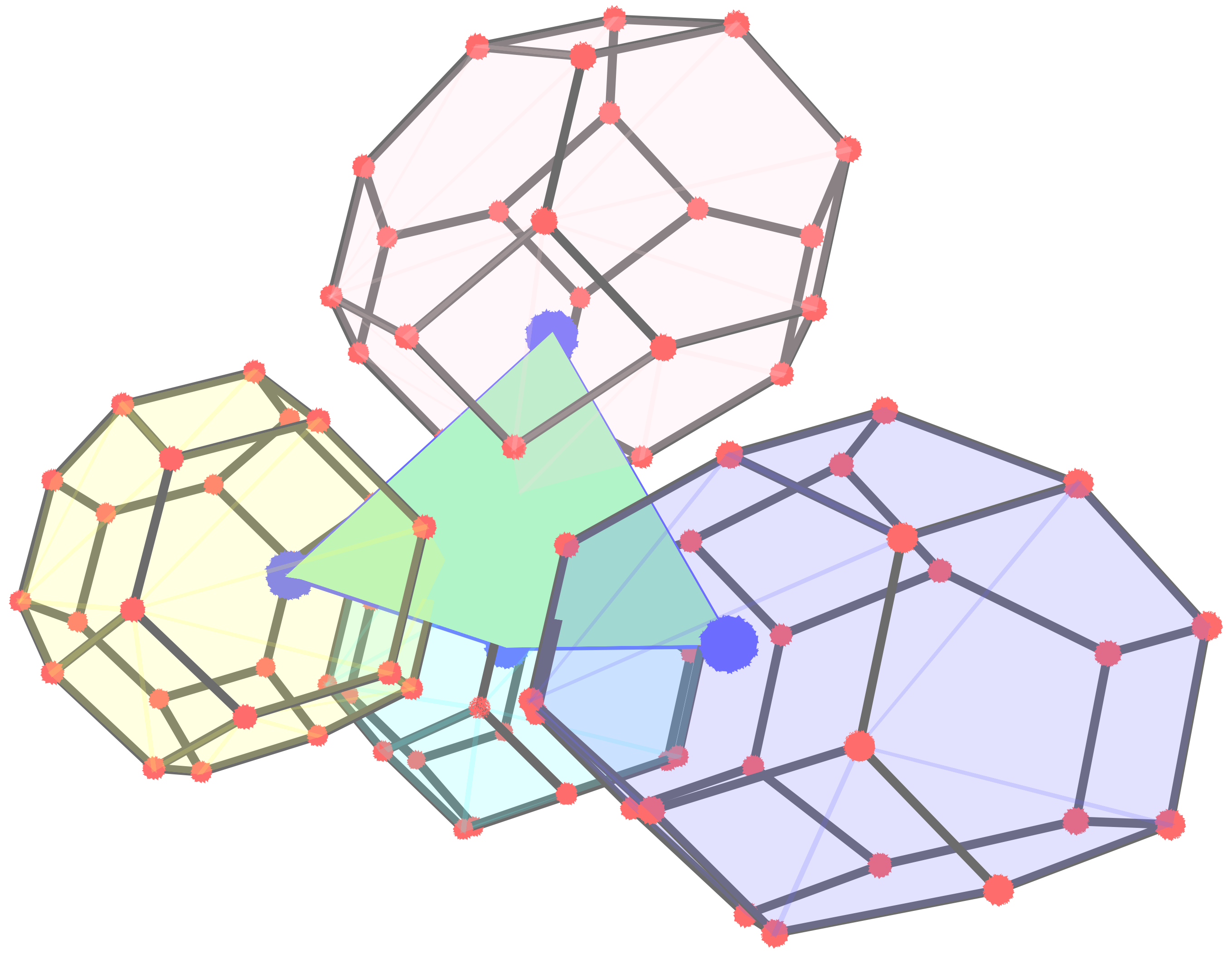
92

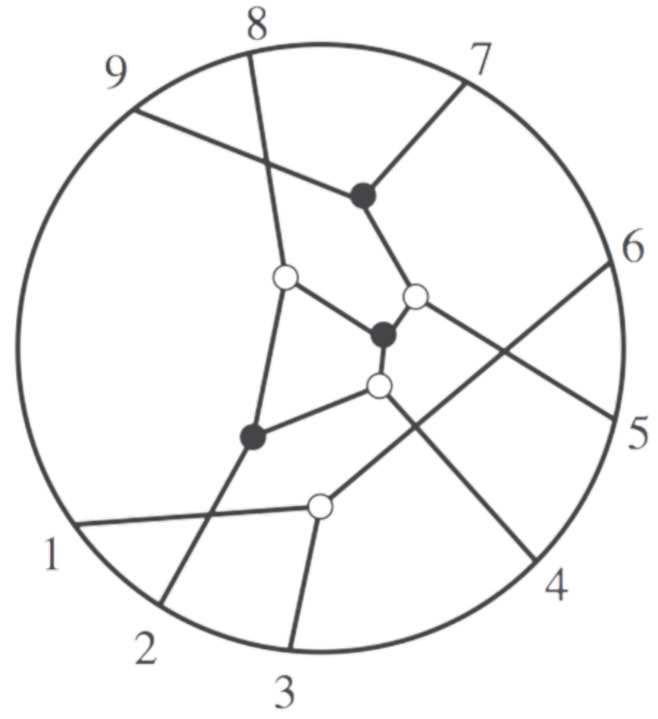




HomotopyContinuation.jl

```
Equations=System(vcat(Point_Conditions,
                      Line_Conditions,
                      Plane_Conditions,
                      det(X)-D, Affine_Chart))
C=certify(Equations,S)
```





Thank you!

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Part 1:

- **KP Solitons from Tropical Limits** (2023). Daniele Agostini, Claudia Fevola, Yelena Mandelshtam, and Bernd Sturmfels. *Journal of Symbolic Computation* **114** 282-301.

- **Hirota Varieties and Rational Nodal Curves** (2022). Claudia Fevola and Yelena Mandelshtam. ArXiv:2203.00203. Submitted to *Journal of Symbolic Computation.*

Part 2:

- **Vector Spaces of Generalized Euler Integrals** (2022). Daniele Agostini, Claudia Fevola, Anna-Laura Sattelberger, and Simon Telen. ArXiv:2208.08967. Submitted to *Communications in Number Theory and Physics.*

- **Likelihood Degenerations** (2023). Daniele Agostini, Taylor Brysiewicz, Claudia Fevola, Lukas Kühne, Bernd Sturmfels, and Simon Telen. *Advances in Mathematics* **414** 108863.

Part 3:

- **Pencils of Quadrics: Old and New** (2021). Claudia Fevola, Yelena Mandelshtam, and Bernd Sturmfels. Le *Matematiche* **76** 319-335.

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